

IMAGES OF GOLOD-SHAFAREVICH ALGEBRAS WITH SMALL GROWTH

LAURENT BARTHOLDI AND AGATA SMOKTUNOWICZ

ABSTRACT. We show that Golod-Shafarevich algebras can be homomorphically mapped onto infinite-dimensional algebras with polynomial growth, under mild assumptions of the number of relations of given degrees.

In case these algebras are finitely presented, we show they can be mapped onto an infinite dimensional algebras with quadratic growth. This answers a question by Zelmanov.

We then show, by an elementary construction, that any sufficiently regular function $\lesssim n^{\log n}$ may occur as the growth of an algebra.

INTRODUCTION

“Is every finitely generated torsion group finite?”

(the *Burnside* problem)

“Is every finitely generated algebraic algebra finite-dimensional?”

(the *Kurosh* problem)

The seminal work of Golod and Shafarevich [3, 4], in 1964, showed that the answer to these famous problems is negative. Their method entailed the construction of an infinite-dimensional finitely generated nil graded algebra R by carefully adding relators; the elements of the form $1 + n$, for n in the generating set of R , generate an infinite torsion group.

Their construction is quite flexible, and has been generalized in various directions, so as to obtain more information on the *growth* of R , which we now define.

Let R be an associative algebra generated by a finite-dimensional subspace S . The *growth* of R is the function $v(n) = \dim(1 + S + S^2 + \cdots + S^n)$. It depends on S , but only mildly: if S' be another generating subspace for R , then the corresponding growth function v' is related to v by inequalities

$$v(n) \leq v'(Cn), \quad v'(n) \leq v(Cn)$$

for some constant C . We write $v \lesssim v'$ and $v \sim v'$ if one, respectively both of the inequalities above are satisfied; then the equivalence class of v is independent of S .

The algebra R has *polynomial* growth if $v(n) \lesssim n^d$ for some d ; the infimal such d is the *Gelfand-Kirillov* dimension of R . It has *superpolynomial* growth if no such d exists. It has *exponential* growth if $v(n) \sim 2^n$, and *subexponential* growth otherwise.

The groups and the algebras constructed by the Golod-Shafarevich method have exponential growth. Much later, Gromov [6] proved that under the assumption that the group has polynomial growth, the answer to the Burnside Problem is positive. In fact, he proved that a finitely generated group with polynomial growth has a

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nilpotent normal subgroup of finite index. As a consequence, if a finitely-generated group has polynomial growth and each element has finite order then the group is finite.

Zelmanov asked in [14, Problem 5] whether Golod-Shafarevich algebras always have an infinitely generated homomorphic images with polynomial growth. It was shown in [11] not to be the case. The same was shown for Golod-Shafarevich groups by Ershov [2] — there exist Golod-Shafarevich groups without infinite images of polynomial growth; indeed there exist Golod-Shafarevich groups satisfying Kazhdan's property (T).

Our first main result is the following

Theorem A (Golod-Shafarevich algebras). *Let \mathbb{K} be an algebraically closed field, and let $A = \mathbb{K}\langle x, y \rangle$ be the free noncommutative algebra generated (in degree one) by elements x, y .*

Let $d \geq 3$ be given; for all $k \in \mathbb{N}$, let a homogeneous subspace $D(k) \leq A(k)$ be given, such that

- $D(k) = 0$ unless $d < (k/2)^{499/500} / \log(5k/2)$;
- $D(k) = 0$ if $2^n - 2^{n-3} < k \leq 2^n + 2^{n-1}$ for some $n \in \mathbb{N}$;
- if $D(k), D(\ell) \neq 0$ and $k < 2^n < \ell$ for some $n \in \mathbb{N}$, then $\ell \geq k^{1000}$;
- $\dim D(k) \leq k^d$ for all $k \in \mathbb{N}$.

Then $A/\langle D(k) : k \in \mathbb{N} \rangle$ can be homomorphically mapped onto an infinite dimensional algebra with Gelfand-Kirillov dimension at most $25d$.

Corollary B (Finitely presented Golod-Shafarevich algebras). *With the same assumptions and notations as in Theorem A, if we assume $D(k) = 0$ for almost all k , then $A/\langle D(k) : k \in \mathbb{N} \rangle$ can be mapped onto an infinite dimensional algebra with quadratic or linear growth.*

This answers a question by Zelmanov [11, and private communication].

We remark that Alexander Young obtained related results, but for special types of ideals with repeated patterns, called *regimented ideals*. For example a regimented ideal generated by single $f \in A$ is of type $\bigcap_{1 \leq i \leq \deg f} \sum_{k \in (\deg f)\mathbb{N}} A(k)fA$. We do not have such restrictions on the ideal I .

More generally, we may wish to construct algebras of prescribed growth, in which a predetermined set of relations have already been imposed. In this sense, we are able to achieve finite Gelfand-Kirillov dimension when the relations are in appropriately separated degrees, and quadratic growth when there are, furthermore, only finitely many relations.

According to Gromov's result mentioned above, if a group has polynomial growth then its growth function is $\sim n^d$ for an integer d . Grigorchuk showed in [5] that there exist semigroups of growth strictly between polynomial and exponential; closely related examples [1] have growth $\exp(n^\alpha)$ for various α , accumulating to 1. One of the tantalizing open problems is the existence of groups of intermediate growth strictly between polynomial and $\exp(n^{1/2})$.

Which functions are the growth function of an associative algebra? An obvious restriction is that the growth function must be submultiplicative (since any $(m+n)$ -fold product of generators can be factored as an m -fold product times an n -fold product). For every real number $\alpha \geq 2$, there exists a finitely generated algebra with Gelfand-Kirillov dimension α ; see [7]. It was also noted by Lenagan [private communication] that, using Warfield's construction [7, pages 19–20], it is possible for functions f that are polynomial at most values of n to construct an algebra with growth between $f(n)$ and $2f(n)$. We address the question of constructing algebras of superpolynomial growth:

Theorem C. *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be submultiplicative and increasing: $f(m+n) \leq f(m)f(n)$ for all m, n , and $f(n+1) \geq f(n)$. Then there exists a finitely generated algebra B whose growth function $v(n)$ satisfies*

$$f(2^n) \leq \dim B(2^n) \leq 4^{n+1} f(2^{n+1}).$$

Furthermore, B may be chosen to be monomial.

Corollary D (Many growth functions). *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be submultiplicative, increasing, and such that $f(Cn) \geq nf(n)$ for some $C > 0$ and all $n \in \mathbb{N}$. Then there exists an associative algebra with growth $\sim f$.*

Note that the hypotheses are satisfied by any sufficiently regular function that grows at least as fast as $n^{\log n}$. The results in [12] therefore hold for a very large class of growth functions. It remains open whether arbitrary functions between polynomial and $n^{\log n}$ can be realized as the growth of an algebra.

Our construction is elementary, and bears resemblance to Zelmanov's construction of a prime algebra with a nonzero locally nilpotent ideal [13]. We do not know if our algebras are prime.

0.1. Sketch of proof of Theorem A. Its proof is an intricate induction, which we broadly explain here. From the subspaces $D(k)$, we proceed as follows:

- in §3, subspaces $F(2^n)$ of $A(2^n)$ are constructed, depending on splittings $U(2^i) \oplus V(2^i) = A(2^i)$ for $i < n$. Roughly speaking, for $k \geq 2^n$, elements of degree 2^{n+1} in $AD(k)A$ are contained in $F(2^n)A(2^n) + A(2^n)F(2^n)$.
- in §1, subspaces $U(2^n), V(2^n) \leq A(2^n)$ are constructed, depending on $U(2^i), V(2^i)$ for $i < n$ and on $F(2^n)$. This part relies heavily on previous results from [8]. Among other properties, they satisfy $V(2^{n-1})^2 \leq V(2^n)$.
- still in §1, for all $k \in \{2^{n-1}, \dots, 2^n - 1\}$, we set

$$\mathcal{E}(k) = \{r \in A(k) \mid ArA \cap A(2^{n+1}) \subseteq U(2^n)A(2^n) + A(2^n)U(2^n)\}$$

and $\mathcal{E} = \bigoplus_k \mathcal{E}(k)$. The desired quotient is then A/\mathcal{E} .

- in §2, we bound the Gelfand-Kirillov dimension of A/\mathcal{E} .

We wrap up the proof of Theorem A and its corollary in §4. We prove Theorem C and its corollary in §5.

0.2. Notation. In what follows, \mathbb{K} is a countable field and A is the free \mathbb{K} -algebra in two non-commuting indeterminates x and y . The set of monomials in $\{x, y\}$ is denoted by M and, for each $k \geq 0$, its subset of monomials of degree k is denoted by $M(k)$. Thus, $M(0) = \{1\}$ and for $k \geq 1$ the elements in $M(k)$ are of the form $x_1 \cdots x_k$ with all $x_i \in \{x, y\}$. The span of $M(k)$ in A is denoted by $A(k)$; its elements are called *homogenous polynomials of degree k* . More generally, for any subset X of A , we denote by $X(k)$ its subset of homogeneous elements of degree k .

The *degree* $\deg f$ of an element $f \in A$ is the least $k \geq 0$ such that $f \in A(0) + \cdots + A(k)$. Any $f \in A$ can be uniquely written in the form $f = f_0 + f_1 + \cdots + f_k$ with each $f_i \in A(i)$. The elements f_i are the *homogeneous components* of f . A (right, left, two-sided) ideal of A is *homogeneous* if it is spanned by its elements' homogeneous components. If V is a linear space over \mathbb{K} , we denote by $\dim V$ the dimension of V over \mathbb{K} . The Gelfand-Kirillov dimension of an algebra R is written $\text{GKdim}(R)$. For elementary properties of Gelfand-Kirillov dimension we refer to [7].

For any real number x , define $\lfloor x \rfloor$ as the largest integer at most x , and $\lceil x \rceil$ as the smallest integer at least x . All logarithms are in base 2.

1. CONSTRUCTING $U(2^n), V(2^n)$

We start with the following result, which is a modification of [8, Theorem 3]. Let subspaces $F(2^n) \subseteq A(2^n)$ be given. Set $Z = \{n \in \mathbb{N} : F(2^n) \neq 0\}$. Moreover, set

$$(1) \quad e(k) = \lfloor \log(5kd) \rfloor,$$

for the given constant d in Theorem A, and

$$(2) \quad S = \bigsqcup_{k \in Z} \{k - e(k) - 1, \dots, k - 1\}$$

(the union is disjoint by the assumption ' $D(k) = 0$ unless $d < (k/2)^{499/500} / \log(5k/2)$ '. Indeed, for $2^n + 2^{n-1} \leq k \leq 2^n + 2^{n-1} + 2^{n-2}$, the set $\{\lfloor k^{1/500} \rfloor, \lfloor k^{1/500} \rfloor + 1, \dots, k\}$ contains the set $\{2^{n-e(n)}, \dots, 2^{n-1}\}$.)

Theorem 1.1. *Let $F(2^n)$, Z and S be as above. Suppose that, for every n , we have $\dim F(2^n) \leq (2^{e(n)})^2 - 2$, and, if $i < j \in Z$ then $10i < j$. Then there are \mathbb{K} -linear subspaces $U(2^n)$ and $V(2^n)$ of $A(2^n)$ such that for all $n \in \mathbb{N}$:*

- (1) $\dim V(2^n) = 2$ if $n \notin S$;
- (2) $\dim V(2^{n-e(n)-1+j}) = 2^{2^j}$ for all $n \in Z$ and all $0 \leq j \leq e(n)$;
- (3) $V(2^n)$ is spanned by monomials;
- (4) $F(2^n) \subseteq U(2^n)$ for every $n \in Z$;
- (5) $V(2^n) \oplus U(2^n) = A(2^n)$;
- (6) $A(2^n)U(2^n) + U(2^n)A(2^n) \subseteq U(2^{n+1})$;
- (7) $V(2^{n+1}) \subseteq V(2^n)V(2^n)$.

Moreover, the definition of $U(2^n)$ and $V(2^n)$ is inductive, and requires only the knowledge of $F(2^m)$ for $m \leq n$.

Proof. Same as in [8]. The only difference is we use $e(n)$ instead of $\log(n)$ in the definition (2) of S . \square

We define then a graded subspace \mathcal{E} of A by constructing its homogeneous components $\mathcal{E}(k)$ as follows. Given $k \in \mathbb{N}$, let $n \in \mathbb{N}$ be such that $2^{n-1} \leq k < 2^n$. Then $r \in \mathcal{E}(k)$ precisely if, for all $j \in \{0, \dots, 2^{n+1} - k\}$, we have $A(j)rA(2^{n+1} - j - n) \subseteq U(2^n)A(2^n) + A(2^n)U(2^n)$. More compactly,

$$(3) \quad \mathcal{E}(k) = \{r \in A(k) \mid ArA \cap A(2^{n+1}) \subseteq U(2^n)A(2^n) + A(2^n)U(2^n)\}.$$

Set then $\mathcal{E} = \bigoplus_{k \in \mathbb{N}} \mathcal{E}(k)$. We recall:

Lemma 1.2 ([8, Theorem 5]). *The set \mathcal{E} is an ideal in A .*

Lemma 1.3 ([8, Theorems 14,15]). *The algebra A/\mathcal{E} is infinite dimensional over \mathbb{K} .*

We now extend the definition of $U(2^n), V(2^n)$ to dimensions that are not powers of 2. The sets (4–7) are called respectively S, W, R, Q in [8, §4].

Let $k \in \mathbb{N}$ be given. Write it as a sum of increasing powers of 2, namely $k = \sum_{i=1}^t 2^{p_i}$ with $0 \leq p_1 < p_2 < \dots < p_t$. Set then

$$(4) \quad U^<(k) = \sum_{i=0}^t A(2^{p_1} + \dots + 2^{p_{i-1}})U(2^{p_i})A(2^{p_{i+1}} + \dots + 2^{p_t}),$$

$$(5) \quad V^<(k) = V(2^{p_1}) \dots V(2^{p_t}),$$

$$(6) \quad U^>(k) = \sum_{i=0}^t A(2^{p_t} + \dots + 2^{p_{i+1}})U(2^{p_i})A(2^{p_{i-1}} + \dots + 2^{p_1}),$$

$$(7) \quad V^>(k) = V(2^{p_t}) \dots V(2^{p_1}).$$

Lemma 1.4 ([8, pp. 993–994]). *For all $k \in \mathbb{N}$ we have $A(k) = U^<(k) \oplus V^<(k) = U^>(k) \oplus V^>(k)$.*

For all $k, \ell \in \mathbb{N}$ we have $A(k)U^<(\ell) \leq U^<(k + \ell)$ and $U^>(k)A(\ell) \leq U^>(k + \ell)$.

These sets are useful to estimate the dimension of A/\mathcal{E} :

Proposition 1.5 ([8, Theorem 11], [9, Theorem 5.2]). *For every $k \in \mathbb{N}$ we have*

$$\dim A(k)/\mathcal{E}(k) \leq \sum_{j=0}^k \dim V^<(k-j) \dim V^>(j).$$

2. THE GELFAND-KIRILLOV DIMENSION OF A/\mathcal{E}

To lighten notation, in this section we write $[X] = \dim X$ for the dimension of a subspace $X \leq A$.

We start with a lemma about the dimensions $V^>(k)$ and $V^<(k)$, continuing on the notation of §1.

Lemma 2.1. *Let α be a natural number, with binary decomposition $\alpha = 2^{p_1} + \dots + 2^{p_t}$. Suppose $p_i \notin S$ for all $i = 1, \dots, t$. Then $[V^>(\alpha)] \leq 2\alpha$.*

Proof. If $p_i \notin S$, then $[V(2^{p_i})] = 2$ by assumption, so

$$[V^>(\alpha)] = \prod_{i=1}^t [V(2^{p_i})] = 2^t \leq 2^{\log(\alpha)+1} \leq 2\alpha. \quad \square$$

Lemma 2.2. *Let α be a natural number, with binary decomposition $\alpha = 2^{p_1} + \dots + 2^{p_t}$. Suppose that there is $k \in Z$ such that $p_i \in \{k - e(k) - 1, \dots, k - 1\}$ for all $i = 1, \dots, t$. Then $[V^>(\alpha)] \leq 2^{10kd}$. More precisely, $[V^>(\alpha)] = 2^{\alpha/2^{k-e(k)-1}}$.*

Proof. Recall that we defined $e(k) = \lfloor \log(5kd) \rfloor$, see (1). Write $S_k = \{k - e(k) - 1, \dots, k - 1\}$. Recall that, by Theorem 1.1(2), we have $[V(2^i)] = 2^{2^{i-(k-e(k)-1)}}$ for all $i \in S_k$. Then

$$\begin{aligned} [V^>(\alpha)] &\leq \prod_{i=0}^{e(k)} [V(2^i)] = \prod_{i=0}^{e(k)} 2^{2^i} = 2^{\sum_{i=0}^{e(k)} 2^i} \leq 2^{2^{e(k)+1}} \\ &\leq 2^{2^{\log(5kd)+1}} = 2^{10kd}. \end{aligned}$$

We now prove $[V^>(\alpha)] = 2^{\alpha/2^{k-e(k)-1}}$. As before, we have

$$\begin{aligned} \log[V^>(\alpha)] &= \log \prod_{i=1}^t [V(2^{p_i})] = \log \prod_{i=1}^t 2^{2^{p_i-(k-e(k)-1)}} \\ &= \sum_{i=1}^t 2^{p_i-(k-e(k)-1)} = \frac{\alpha}{2^{k-e(k)-1}}. \end{aligned} \quad \square$$

Proposition 2.3. *Let α be a natural number. Then $[V^>(\alpha)] < 2\alpha^{12d}$.*

Proof. Write $\alpha = 2^{p_1} + \dots + 2^{p_t}$ in binary. Write again $S_k = \{k - e(k) - 1, \dots, k - 1\}$. For all $k \in \mathbb{N}$, set $\alpha_k = \sum_{p_i \in S_k} 2^{p_i}$. Set $\gamma = \sum_k \alpha_k$ and $\delta = \sum_{p_i \notin S} 2^{p_i}$, so that $\alpha = \gamma + \delta$. By definition of the sets $V^>(m)$, we have $[V^>(\alpha)] = [V^>(\gamma)][V^>(\delta)]$. By Lemma 2.2 we have $[V^>(\alpha_k)] \leq 2^{10kd}$ for all k .

Note now that, by the assumptions of Theorem A, if $k < k' \in Z$ then $500k < k'$. Let $m \in \mathbb{N}$ be maximal such that $\alpha_m \neq 0$. We deduce

$$[V^>(\gamma)] = \prod_{k \leq m, k \in Z} [V^>(\alpha_k)] < \prod_{i \in \mathbb{N}} 2^{10md/500^i} \leq 2^{10md/500/499}.$$

Moreover, from the binary form of α , we get $\alpha < 2^{m+1}$, so $[V^>(\gamma)] \leq \alpha^{11d}$.

Finally, by Lemma 2.1, we have $[V^>(\delta)] \leq 2\alpha$. Putting everything together, we get $[V^>(\alpha)] < 2\alpha^{12d}$. \square

Lemma 2.4. *Let α, β be natural numbers such that $\alpha + \beta \leq 2^{n-1} + 2^{n-2}$ for some $n \in \mathbb{Z}$. Then*

$$[V^<(\alpha)][V^>(\beta)] \leq \frac{1}{2^{(n+1)(d+2)+2}} [V(2^{n-1})]^2.$$

Proof. Write $\alpha = 2^{p_1} + \dots + 2^{p_t}$ in binary. Write again $S_k = \{k - e(k) - 1, \dots, k - 1\}$ and $\alpha_k = \sum_{p_i \in S_k} 2^{p_i}$. Set now $\gamma = \sum_{k < n} \alpha_k$ and $\delta = \sum_{p_i \notin S} 2^{p_i}$; we get $\alpha = \gamma + \delta + \alpha_n$, and by definition of the sets $V^>(n)$ we get $[V^>(\alpha)] = [V^>(\gamma)][V^>(\delta)][V^>(\alpha_n)]$.

As in Proposition 2.3,

$$[V^>(\gamma)] = \prod_{k < n/500, k \in \mathbb{Z}} [V^>(\alpha_k)] < \prod_{i \in \mathbb{N}} 2^{10dp_t/500^{i+1}} \leq 2^{10dp_t/499} \leq \alpha^{10d/499}.$$

By Lemma 2.1, we get

$$[V^>(\delta)] \leq 2\delta \leq 2\alpha.$$

By Lemma 2.2, we get

$$[V^>(\alpha_n)] = 2^{\alpha_n/2^{n-e(n)-1}} \leq 2^{\alpha/2^{n-e(n)-1}}.$$

Therefore,

$$[V^>(\alpha)] \leq 2\alpha^{1+10d/499} 2^{\alpha/2^{n-e(n)-1}}.$$

By the definition of sets $V^<$ and $V^>$, we get $[V^<(\alpha)] = [V^>(\alpha)]$, so

$$[V^<(\alpha)][V^>(\beta)] \leq 4(\alpha\beta)^{1+10d/499} 2^{\frac{\alpha+\beta}{2^{n-e(n)-1}}}.$$

Since $\alpha + \beta \leq 2^{n-1} + 2^{n-2}$ so $\alpha\beta \leq 2^{2n-2}$, we get

$$\begin{aligned} \log([V^<(\alpha)][V^>(\beta)]) &\leq (2n-2)(1+10d/499) + 2 + \frac{2^{n-1} + 2^{n-2}}{2^{n-e(n)-1}} \\ &= (2n-2)(1+10d/499) + 2 + 2^{e(n)} + 2^{e(n)-1}. \end{aligned}$$

By Theorem 1.1(2) we have $[V(2^{n-1})] = 2^{2^{e(n)}}$, so

$$\begin{aligned} \log([V^>(\alpha)][V^<(\beta)]) &\leq (2n-2)(1+10d/499) + 2 + \log([V(2^{n-1})]^2) - 2^{e(n)-1} \\ &\leq (2n-2)(1+10d/499) + 2 + \log([V(2^{n-1})]^2) - 5dn \\ &\leq \log([V(2^{n-1})]^2) - ((n+1)(d+2) + 2) \end{aligned}$$

as required. The last inequality holds thanks to our assumption $d \geq 3$. \square

Lemma 2.5. *Let $F(2^n), U(2^n), V(2^n), S$ be as in Theorem 1.1. Let \mathcal{E} be defined as in (3). Then the algebra A/\mathcal{E} has Gelfand-Kirillov dimension at most $25d$.*

Proof. By Proposition 1.5, we have $\dim(A(k)/\mathcal{E}(k)) \leq \sum_{j=0}^k [V^<(k-j)][V^>(j)]$. By Proposition 2.3 we have $\dim(A(k)/\mathcal{E}(k)) \leq \sum_{j=0}^k 2k^{24d} \leq 2k^{25d}$. Therefore, $\text{GKdim}(A/\mathcal{E}) \leq 25d$. \square

3. CONSTRUCTING $F(2^n)$

In this section, we construct the sets $F(2^n) \leq A(2^n)$ that let us apply Theorem 1.1. We proceed by steps:

Lemma 3.1. *Let the notation be as in Theorem 1.1 and Theorem A. Consider all $D(k) \leq A(k)$ with $2^n + 2^{n-1} \leq k \leq 2^n + 2^{n-1} + 2^{n-2}$. Suppose we defined sets $U(2^m) \leq A(2^m)$ for all $m < n$, and suppose $n \in \mathbb{Z}$.*

Then there exists a linear \mathbb{K} -space $F'(2^n) \subseteq A(2^n)$ with the following properties:

- $\dim F'(2^n) \leq \frac{1}{2} \dim V(2^{n-1})^2$;
- for all $i, j \geq 0$ with $i + j = k - 2^n$ we have $D(k) \subseteq A(i)F'(2^n)A(j) + U^{<}(i)A(k-i) + A(k-j)U^{>}(j)$, with the sets $U^{<}(i), U^{>}(i)$ defined in (4- γ).

Proof. Consider $f \in D(k)$. We can write f in the form $f = \tilde{f} + g$, with $g \in U^{<}(i)A(k-i) + A(k-j)U^{>}(j)$ and

$$\tilde{f} = \sum_{c \in V^{<}(i), d \in V^{>}(j)} cz_{c,d,f}d, \quad z_{c,d,f} \in A(2^n).$$

Still for that given f , we restrict the c, d above to belong to a basis, and let $T(i, j, f) \leq A(2^n)$ be the subspace spanned by all the $z_{c,d,f}$ above. We then have $\dim T(i, j, f) \leq \dim V^{<}(i) \dim V^{>}(j)$. Observe also $f \in A(i)T(i, j, f)A(j) + U^{<}(i)A(k-i) + A(k-j)U^{>}(j)$, because $U^{<}(i) \oplus V^{<}(i) = A(i)$ and $A(j) = U^{>}(j) \oplus U^{>}(j)$. Define

$$F'(2^n) = \sum_{k=2^n+2^{n-1}}^{2^n+2^{n-1}+2^{n-2}} \sum_{f \in D(k)} \sum_{i+j=k-2^n} T(i, j, f).$$

We have $2^{n-1} \leq i + j \leq 2^{n-1} + 2^{n-2}$, so

$$\dim F'(2^n) \leq 2^{2n-2} 2^{(n+1)d} \sup_{2^{n-1} \leq i+j \leq 2^{n-1}+2^{n-2}} \dim V^{<}(i) \dim V^{>}(j) \leq \frac{1}{2} \dim V(2^{n-1})^2$$

by Lemma 2.4. \square

Lemma 3.2. *Let R be a commutative finitely generated algebra of Gelfand-Kirillov dimension t . Let I be a principal homogeneous ideal in A , that is, an ideal generated by one homogeneous element. Then $Q = A/I$ has Gelfand-Kirillov dimension at least $t - 1$.*

Proof. Write $I = cA$ for some $c \in A$. Then as A and A/I graded we have $Q(n) = A(n)/I(n) = A(n)/cA(n - \deg c)$, so $\dim Q(n) = \dim A(n) - \dim A(n - \deg c)$. Suppose for contradiction $\text{GKdim } Q < t - 1$, so $\text{GKdim } Q = t - 1 - \epsilon$ for some $\epsilon > 0$. Consider $q \in (t - 1 - \epsilon, t - 1)$. Then, by the definition of Gelfand-Kirillov dimension, we have $\dim Q(n) < n^q$ for almost all n , so there is $C \in \mathbb{R}$ such that $\dim Q(n) < Cn^q$ for all n . Observe now, for all $k \in \mathbb{N}$, that

$$\begin{aligned} \dim A(k \deg c) &= \sum_{i=1}^k \dim A(i \deg c) - \dim A((i-1) \deg c) \\ &= \sum_{i=1}^k \dim Q(i \deg c) < \sum_{i=1}^k C(i \deg c)^q < Ck(k \deg c)^q, \end{aligned}$$

so $\text{GKdim } A \leq q + 1$, a contradiction with $q < t - 1$. \square

We are now ready to construct the space $F(2^n)$.

Proposition 3.3. *With notation as above, there is a linear \mathbb{K} -space $F(2^n) \subseteq A(2^n)$ containing $F'(2^n)$ and satisfying $\dim F(2^n) \leq \dim V(2^{n-1})^2 - 2$. Moreover, for all $k \in \{2^n + 2^{n-1}, \dots, 2^n + 2^{n-1} + 2^{n-2}\}$ we have*

$$\begin{aligned} AD(k)A \cap A(2^{n+1}) &\subseteq A(2^n)F(2^n) + F(2^n)A(2^n) \\ &\quad + U(2^{n-1})A(2^n + 2^{n-1}) + A(2^{n-1})U(2^{n-1})A(2^n) \\ &\quad + A(2^n)U(2^{n-1})A(2^{n-1}) + A(2^n + 2^{n-1})U(2^{n-1}). \end{aligned}$$

Proof. Choose a \mathbb{K} -linear subspace $C \leq V(2^{n-1})V(2^{n-1})$ such that

$$(8) \quad C \oplus (F'(2^n) + U(2^{n-1})A(2^{n-1}) + A(2^{n-1})U(2^{n-1})) = A(2^n).$$

By Lemma 3.1, we may choose a basis $\{c_1, \dots, c_s\}$ of C with

$$(9) \quad s = \dim C \geq \dim V(2^{n-1})V(2^{n-1}) - \dim F'(2^n) \geq \frac{1}{2} \dim V(2^{n-1})^2.$$

Let $R = \mathbb{K}[y_1, \dots, y_s, z_1, \dots, z_s]$ be the ring of polynomials in $2s$ indeterminates, and let Y, Z be two non-commuting indeterminates over R . Define a \mathbb{K} -linear map $\Phi: C \rightarrow RY + RZ$ by

$$\Phi(c_t) = y_t Y + z_t Z \quad \text{for } i = 1, \dots, s.$$

Using (8), extend Φ to a \mathbb{K} -linear map $A(2^n) \rightarrow RY + RZ$ by the condition $\ker(\Phi) = F'(2^n) + U(2^{n-1})A(2^{n-1}) + A(2^{n-1})U(2^{n-1})$.

Consider now $f \in D(k)$, with $k \in \{2^n + 2^{n-1}, \dots, 2^n + 2^{n-1} + 2^{n-2}\}$. Consider also $i, j \in \mathbb{N}$ such that $i + j + k = 2^{n+1}$, so we have $2^{n-2} \leq i + j \leq 2^{n-1}$. Consider furthermore $b \in A(i)$ and $d \in A(j)$. Then $bfd \in CC + \ker(\Phi)A(2^n) + A(2^n)\ker(\Phi)$. It follows that there are $\alpha_{bfd}^{t,u} \in \mathbb{K}$ such that

$$bfd = \sum_{1 \leq t, u \leq s} \alpha_{bfd}^{t,u} c_t c_u \mod \ker(\Phi)A(2^n) + A(2^n)\ker(\Phi).$$

Define now a \mathbb{K} -linear map $\Psi: A(2^{n+1}) \rightarrow RYY + RYZ + RZY + RZZ$ by

$$\Psi(c_t c_u) = \Phi(c_t)\Phi(c_u), \quad \ker(\Psi) = A(2^n)\ker(\Phi) + \ker(\Phi)A(2^n).$$

We get $\Psi(bfd) = \Psi(\sum_{1 \leq t, u \leq s} \alpha_{bfd}^{t,u} c_t c_u)$ and so

$$\Psi(bfd) = p_{bfd}^{YY}YY + p_{bfd}^{YZ}YZ + p_{bfd}^{ZY}ZY + p_{bfd}^{ZZ}ZZ$$

for some polynomials $p_{bfd}^{YY}, \dots, p_{bfd}^{ZZ} \in R$. Recall the sets $V^>(i)$, $V^<(j)$ from §1. Define a \mathbb{K} -linear subspace E of R as follows:

$$E = \sum_{k=2^n+2^{n-1}}^{2^n+2^{n-1}+2^{n-2}} \sum_{f \in D(k)} \sum_{i+j=2^{n+1}-k} \sum_{b \in A(i), d \in A(j)} \mathbb{K}p_{bfd}^{YY} + \dots + \mathbb{K}p_{bfd}^{ZZ}.$$

By Lemma 2.4, the inner sum has dimension at most $4 \dim V(2^{n-1})^2 / 2^{(n+1)(d+2)+2}$. Summing over all $i + j = 2^{n+1} - k$ multiplies by a factor of 2^{n+1} at most; summing over all $f \in D(k)$ multiplies by a factor of $\dim D(k) \leq k^d \leq 2^{(n+1)d}$ at most; and summing over all k multiplies by a factor of 2^{n-1} at most. Therefore,

$$\begin{aligned} \dim E &\leq 2^{n-1} 2^{(n+1)d} 2^{n+1} 4 \dim V(2^{n-1})^2 / 2^{(n+1)(d+2)+2} \\ &\leq \frac{1}{4} \dim V(2^{n-1})^2 \leq \frac{1}{2} \dim C \leq s - 2 \quad \text{by (9).} \end{aligned}$$

We will show that there are $\eta_t, \zeta_t \in \mathbb{K}$ for $t = 1, \dots, s$ such that $\Psi(t)(\{\eta_t, \zeta_t\}) = 0$ for all $t \in E$; namely, if we substitute $y_t := \eta_t$ and $z_t := \zeta_t$ in $\Psi(t)$, we get 0. Moreover, we will find $c_u, c_v \in C$ such that $\Phi(c_u), \Phi(c_v) \in RY + RZ$ are linearly independent over R .

We proceed by contradiction. Assume that all assignments $y_t = \eta_t$, $z_t = \zeta_t$ satisfying $\Psi(E) = 0$ also satisfy $\eta_u \zeta_v - \zeta_u \eta_v = 0$ for all $u, v \in \{1, \dots, s\}$. By Hilbert's Nullstellensatz, the polynomials $y_t z_u - z_t y_u$ vanish on all common zeros of $\Psi(E)$, and so there is $m \in \mathbb{N}$ such that $(y_t z_u - z_t y_u)^m \in R\Psi(E)$. It follows that $R/R\Psi(E)$ has Gelfand-Kirillov dimension at most $s + 1$: it is a finite-dimensional module over $\sum_{X \subset \{y_1, \dots, y_s, z_1, \dots, z_s\} : \#X = s+1} \mathbb{K}[X]$.

On the other hand, by applying $\dim(E)$ times Lemma 3.2, we see that the dimension of $R/R\Psi(E)$ is at least $2s - \dim E$. Since $\dim(E) \leq s - 2$, we have

reached a contradiction. It follows we can find a desired solution $\{\eta_t, \zeta_t\}_{t=1, \dots, s}$, and indices u, v such that $\eta_u \zeta_v - \zeta_u \eta_v \neq 0$.

Define now a \mathbb{K} -linear mapping $\overline{\Phi}: C \rightarrow \mathbb{K}Y + \mathbb{K}Z$ by $\overline{\Phi}(c_t) = \eta_t Y + \zeta_t Z$, and extend it as before to $A(2^n)$ by $\ker \overline{\Phi} = \ker \Phi$. Then $\overline{\Phi}(c_u) := \eta_u Y + \zeta_u Z$ and $\overline{\Phi}(c_v) := \eta_v Y + \zeta_v Z$ give two elements that are linearly independent over \mathbb{K} . Define finally

$$F(2^n) = \ker \overline{\Phi}.$$

By construction, $\overline{\Phi}(E) = 0$, so $E \leq F(2^n)$. Therefore, $AD(k)A \cap A(2^{n+1}) \subseteq A(2^n)F(2^n) + F(2^n)A(2^n) + \sum_{i=0}^3 A(2^{n-1}i)U(2^{n-1})A(2^{n-1}(3-i))$, as required, and $\dim F(2^n) \leq \dim V(2^{n-1})^2 - 2$ because $\overline{\Phi}(c_u), \overline{\Phi}(c_v)$ are linearly independent over \mathbb{K} . Finally $F'(2^n)$ is contained in $F(2^n)$, because $F'(2^n)$ belongs to the kernel of $\overline{\Phi}$. \square

4. PROOF OF THEOREM A AND COROLLARY B

We are now ready to prove our main result.

Theorem 4.1. *The set \mathcal{E} defined in (3) is an ideal in A . Moreover A/\mathcal{E} is an algebra of Gelfand-Kirillov dimension at most $25d$, which is infinite dimensional over \mathbb{K} , and in which $D(k) = 0$ for all k .*

Moreover if almost all sets $D(k)$ are zero, then A/\mathcal{E} has quadratic or linear growth.

Proof. Construct simultaneously sets $F(2^n)$ using Proposition 3.3 and sets $U(2^n), V(2^n)$ using Theorem 1.1. Consider now $k \in \mathbb{N}$ with $2^n < k < 2^{n+1}$. We claim that $D(k)$ is contained in \mathcal{E} ; to see that, it suffices to check $A(i)D(k)A(j) \subset T := A(2^{n+1})U(2^{n+1}) + U(2^{n+1})A(2^{n+1})$ for all $i, j \in \mathbb{N}$ with $i + j + k = 2^{n+2}$.

If $i \geq 2^{n+1}$, we may apply Proposition 3.3 to get $A(i - 2^{n+1})D(k)A(j) \leq A(2^n)F(2^n) + F(2^n)A(2^n) + U(2^{n+1})$ so $A(i)D(k)A(j) \leq T$. Similarly, if $j \geq 2^{n+1}$, we get $A(i)D(k)A(j - 2^{n+1}) \leq U(2^{n+1})$ so $A(i)D(k)A(j) \leq T$. If $i, j \geq 2^n$ then $A(i - 2^n)D(k)A(j - 2^n) \leq A(2^n)U(2^n) + U(2^n)A(2^n)$ so $A(i)D(k)A(j) \leq T$.

If $i < 2^n$ and $j < 2^{n+1}$, then $D(k) \leq A(2^n - i)F'(2^n)A(2^{n+1} - j) + U^{<}(2^n - i)A(2^{n+1} + 2^n - j) + A(2^{n+1} - i)U^{>}(2^{n+1} - j)$ by Lemma 3.1, so $A(i)D(k)A(j) \leq A(2^n)F'(2^n)A(2^{n+1}) + T \leq T$. The case $i < 2^{n+1}, j < 2^n$ is handled similarly.

We may now conclude that $D(k) = 0$ holds in A/\mathcal{E} . By Lemma 2.5, the dimension of $\dim A(n)/\mathcal{E}(n)$ is at most $25d$.

For the second claim of the theorem, assume that almost all $D(k)$ are zero, namely $D(k) = 0$ for all $n > t$.

Then, in Theorem 1.1, for all $n > t$ we only need consider case **2**. Therefore, we can add an assumption in the same manner as in [9] or in [10] that, for each $n > t$, there are $m_1, m_2 \in A(2^n)$ such that $V(2^n) = \mathbb{K}m_1 + \mathbb{K}m_2$ and $m_1 m_1, m_1 m_2 \in U(2^{n+1})$, and define sets $V^{>}(i), V^{<}(j)$ as in [9]. Then, as in [9] we slightly modify the definitions of the sets $V^{>}(i), V^{<}(j)$ to obtain that the algebra has quadratic growth, because there is constant C such that $\dim V^{<}(i) < c$ and $\dim V^{>}(j) < c$ for all i, j . \square

5. GROWTH OF ALGEBRAS

We prove Theorem C in this section. First, we write $d = f(1)$, and note that $f(n) \leq d^n$ follows from submultiplicativity. We will construct a d -generated monomial algebra B with growth approximately f , as a quotient of the free algebra $A = \mathbb{K}\langle x_1, \dots, x_d \rangle$.

Let $M(n)$ denote the set of monomials in A of degree n , and set $M = \bigcup_{n \geq 0} M(n)$. We construct subsets $W(2^n)$ of monomials in $M(2^n)$, inductively as follows. Firstly, $M(1) = W(1) = \{x_1, \dots, x_d\}$. Assuming $W(2^{n-1})$ has been constructed, let

$C(2^n)$ be an arbitrary subset of $W(2^n)$ of cardinality $\lceil f(2^{n+1})/f(2^n) \rceil$. Define then $W(2^{n+1}) = C(2^n)W(2^n)$. Set $W = \bigcup_{n \geq 0} W(2^n)$. Finally, let

$$B = A/\langle w \in M \mid AwA \cap W = \emptyset \rangle$$

be the monomial algebra with relators all words that are not subwords of some word in W .

Since B is a monomial algebra, its growth is computed by estimating the number of non-zero monomials of given length in B . We do this at powers of 2.

Lemma 5.1. *The set W is linearly independent in B .*

Proof. In a monomial algebra, monomials are linearly independent as soon as they are distinct and nonzero. If $w \in W$ were 0 in B , we would have $w = avb$ for some $v \in M$ such that $AvA \cap W = \emptyset$; this contradicts $w \in W$. \square

Lemma 5.2. *Let $w \in M$ be a word of degree 2^m . Assume that w is a subword of $C(2^n)W(2^n)$ or of $W(2^n)C(2^n)$ for some $n > m$. Then w is a subword of $C(2^{n-1})W(2^{n-1})$ or of $W(2^{n-1})C(2^{n-1})$.*

Proof. Let w be a subword of some word $u \in W(2^n)C(2^n) \cup C(2^n)W(2^n)$; write $u = u_1u_2$ with $u_1 \in W(2^{n-1})$ and $u_2 \in C(2^{n-1})$. If w is a subword of u_1 or of u_2 , then w is a subword of a word in $W(2^n)$, since $C(2^n) \subset W(2^n)$. Because $W(2^n) = C(2^{n-1})W(2^{n-1})$, we are done.

If w overlaps u_1 and u_2 , write $u_1 = u_{11}u_{12}$ and $u_2 = u_{21}u_{22}$; then $u_{21} \in C(2^{n-1})$ because $u_2 \in W(2^n)$. By assumption, $n-1 \geq m$, so u is a subword of $u_{12}u_{21}$, which belongs to $W(2^{n-1})C(2^{n-1})$ as required. \square

Lemma 5.3. *Every non-zero degree- 2^m monomial in B is a subword of a monomial in $W(2^m)C(2^m) \cup C(2^m)W(2^m)$.*

Proof. Let $w \in M(2^m)$ be non-zero; so $awb \in W(2^n) = C(2^{n-1})W(2^{n-1})$ for some $n \geq m$. Apply then $m-n-1$ times Lemma 5.2. \square

Lemma 5.4. *For all $n \in \mathbb{N}$, we have*

$$f(2^n) \leq \#W(2^n) < 2^n f(2^n).$$

Proof. By induction; $\#W(1) = f(1)$, and $f(2^{n+1}) \leq f(2^n)\#C(2^n) < f(2^{n+1}) + f(2^n)$, so

$$f(2^{n+1}) \leq \#W(2^{n+1}) = \#W(2^n)\#C(2^n) < 2^n(f(2^{n+1}) + f(2^n)) \leq 2^{n+1}f(2^{n+1}). \quad \square$$

Proof of Theorem C. By Lemmata 5.1 and 5.4, we have

$$\dim B(2^n) \geq \#W(2^n) \geq f(2^n).$$

By Lemma 5.4, the cardinality of $C(2^n)W(2^n)$ is less than $2^n(f(2^{n+1}) + f(2^n))$; and similarly for $W(2^n)C(2^n)$. Each of these monomials has at most $2^n + 1$ distinct subwords of length 2^n . Therefore, by Lemma 5.3,

$$\dim B(2^n) \leq 2(2^n + 1)\#W(2^n)\#C(2^n) < 2^{n+1}(2^n + 1)(f(2^{n+1}) + f(2^n)). \quad \square$$

Lemma 5.5. *If f, g be two increasing functions such that $f(2^n) \leq g(2^n)$ holds for all n , then $f \lesssim g$.*

Proof. For any $m \in \mathbb{N}$, let $n \in \mathbb{N}$ be minimal such that $m \leq 2^n$. We have $f(m) \leq f(2^n) \leq g(2^n) \leq g(2m)$, so $f \lesssim g$. \square

Proof of Corollary D. Let f be a submultiplicative, increasing function with $f(Cn) \geq nf(n)$. Note that this implies $f(n) \sim nf(n)$. By Theorem C and Lemma 5.5, there exists an algebra B with $\dim B(n) \sim f(n)$. Again using $f(n) \sim nf(n)$, the growth of B satisfies $v(n) \sim f(n)$. \square

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GEORG-AUGUST UNIVERSITÄT ZU GÖTTINGEN

E-mail address: Laurent.Bartholdi@gmail.com

MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES, SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH, JAMES CLERK MAXWELL BUILDING, KING'S BUILDINGS, MAYFIELD ROAD, EDINBURGH EH9 3JZ, SCOTLAND, UK

E-mail address: A.Smoktunowicz@ed.ac.uk